

# Spontaneous $\mathcal{PT}$ -symmetry breaking for systems of noncommutative Euclidean Lie algebraic type

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**ABSTRACT:** We propose a noncommutative version of the Euclidean Lie algebra  $E_2$ . Several types of non-Hermitian Hamiltonian systems expressed in terms of generic combinations of the generators of this algebra are investigated. Using the breakdown of the explicitly constructed Dyson maps as a criterium, we identify the domains in the parameter space in which the Hamiltonians have real energy spectra and determine the exceptional points signifying the crossover into the different types of spontaneously broken  $\mathcal{PT}$ -symmetric regions with pairs of complex conjugate eigenvalues. We find exceptional points which remain invariant under the deformation as well as exceptional points becoming dependent on the deformation parameter of the algebra.

## 1. Introduction

In [1] we demonstrated that analogues of quasi-solvable models of Lie algebraic type can be constructed in terms of Euclidean Lie algebra generators. Unlike standard quasi-solvable models, this type of systems admits solutions that can not be expressed in terms of hypergeometric functions. Thus they constitute a different type of class than the more common  $sl_2(\mathbb{C})$ -models [2] with their compact and non-compact real forms  $su(2)$  and  $su(1, 1)$  [3, 4]. We identified various types of  $\mathcal{PT}$ -symmetries for the  $E_2$ -algebra, which for concrete non-Hermitian models served to explain the reality of their spectra in part of the parameter space. Similar features were also previously observed for special cases of the  $E_2$  [5] and  $E_3$  [6] Euclidean Lie algebra. Further interest in these kind of models stems from the fact that for specific representations the models become identical to some complex potential systems currently investigated in optics [7, 8, 9, 10, 11, 12] and solid state physics [13].

Here we continue our investigations by considering deformations of the systems studied in [1]. In particular, we aim to identify the regions in the parameter space where the models possess real eigenvalue spectra due to intact  $\mathcal{PT}$ -symmetry and spontaneously broken  $\mathcal{PT}$ -symmetry where the eigenvalue spectra contain at least one pair of complex conjugate

eigenvalues, characterized by  $[\mathcal{PT}, H] = 0$ ,  $\mathcal{PT}\psi = \psi$  and  $[\mathcal{PT}, H] \neq 0$ ,  $\mathcal{PT}\psi \neq \psi$ , respectively [14, 15, 16]. The transition from one to the other region is marked by the so-called exceptional points [17, 18, 19, 20].

Alternatively, the two different regions are also characterized by a break down of the so-called Dyson map  $\eta$ , defined as the map that adjointly maps a non-Hermitian Hamiltonian  $H \neq H^\dagger$  to isospectral Hermitian counterparts  $h = h^\dagger$  by means of  $h = \eta H \eta^{-1}$  [21, 22]. We will exploit here the latter criterium without computing explicit solutions to the Schrödinger equation, apart from some exceptional cases.

## 2. Deformations of the Euclidean $E_2$ -Lie algebra and their $\mathcal{PT}$ -symmetries

We commence by introducing some natural deformations of the Euclidean Lie algebra  $E_2$ , whose defining commutation relations for their three generators  $u, v$  and  $j$  are

$$[u, j] = iv, \quad [v, j] = -iu, \quad \text{and} \quad [u, v] = 0. \quad (2.1)$$

Some representations for this algebra may be found for instance in [1]. A useful version for our purposes here, not reported in [1], is the two-dimensional one

$$j := yp_x - xp_y, \quad u := x, \quad \text{and} \quad v := y, \quad (2.2)$$

expressed in terms of generators of the Heisenberg canonical variables  $x, y, p_x, p_y$  with non-vanishing commutators  $[x, p_x] = [y, p_y] = i$ . We have set the reduced Planck constant to  $\hbar = 1$ . We can now simply consider various two-dimensional canonical spaces and investigate how they are translated into the  $E_2$ -setting. The most evident choice would be to replace the canonical variables in (2.2) with some variable on a flat noncommutative space. In general, that does not lead to a closure of the algebra. However, when we carry out a Bopp-shift only in the  $u$  and  $v$  generators, we obtain the new generators

$$J := yp_x - xp_y, \quad U := x - \frac{\theta}{2}p_y, \quad \text{and} \quad V := y + \frac{\theta}{2}p_x, \quad (2.3)$$

obeying the deformed  $E_2$ -algebra

$$[U, J] = iV, \quad [V, J] = -iU, \quad \text{and} \quad [U, V] = i\theta. \quad (2.4)$$

We notice that the deformed  $E_2$ -algebra is left invariant with regard to following antilinear maps [14] reported for the  $E_2$ -algebra in [1]

$$\begin{aligned} \mathcal{PT}_3 : J &\rightarrow J, U \rightarrow V, \quad V \rightarrow U, \quad i \rightarrow -i, \\ \mathcal{PT}_4 : J &\rightarrow J, U \rightarrow -U, V \rightarrow V, \quad i \rightarrow -i, \\ \mathcal{PT}_5 : J &\rightarrow J, U \rightarrow U, \quad V \rightarrow -V, i \rightarrow -i. \end{aligned} \quad (2.5)$$

We also observe that the additional symmetries, which hold for the original  $E_2$ -algebra (2.1),  $\mathcal{PT}_1 : j \rightarrow -j, u \rightarrow -u, v \rightarrow -v, i \rightarrow -i$  and  $\mathcal{PT}_2 : j \rightarrow -j, u \rightarrow u, v \rightarrow v, i \rightarrow -i$ , are broken for the deformed version (2.4). While these symmetries are completely excluded here, it was found in [1] that they lead to less interesting models in the undeformed case as the considered general Hamiltonians in the  $E_2$ -generators become Hermitian without any further transformation needed by implementing the constraints discussed therein.

### 3. $\mathcal{PT}$ -symmetric and spontaneously broken regions in parameter space

Let us now indicate how we distinguish the  $\mathcal{PT}$ -symmetric regions in parameter space from the spontaneously broken  $\mathcal{PT}$ -symmetric ones by analyzing the properties of the Dyson map. We start from non-Hermitian  $\mathcal{PT}_{3/4/5}$ -invariant Hamiltonians  $H$  in term of generators of the deformed  $E_2$ -algebra (2.4). We consider all Hamiltonians of this type that may be brought into the general form

$$H_{\mathcal{PT}}(U, V, J) = \hat{c}_1 J^2 + \hat{c}_2 J + \hat{c}_3 U + \hat{c}_4 V + \hat{c}_5 UJ + \hat{c}_6 VJ + \hat{c}_7 U^2 + \hat{c}_8 V^2 + \hat{c}_9 UV + \hat{c}_{10} \quad (3.1)$$

with  $\hat{c}_j = \hat{\alpha}_j + i\hat{\beta}_j$  for  $j = 1, \dots, 10$  and  $\hat{\alpha}_j, \hat{\beta}_j \in \mathbb{R}$ . The specific form of the constants  $\hat{c}_j$  of being either real or purely imaginary is governed by the particular  $\mathcal{PT}$ -symmetry we wish to implement. Similarly as in [1] we act adjointly with the so-called Dyson map on the Hamiltonian in (3.1) and demand that the resulting expression is Hermitian, that is  $\eta H \eta^{-1} = h = h^\dagger$ . The transformed Hamiltonian will be of the same general form as  $H_{\mathcal{PT}}$

$$h_{\mathcal{PT}}(U, V, J) = c_1 J^2 + c_2 J + c_3 U + c_4 V + c_5 UJ + c_6 VJ + c_7 U^2 + c_8 V^2 + c_9 UV + c_{10}, \quad (3.2)$$

albeit with different constants  $c_j = \alpha_j + i\beta_j$  for  $j = 1, \dots, 10$  and  $\alpha_j, \beta_j \in \mathbb{R}$ . Computing then  $h_{\mathcal{PT}} = h_{\mathcal{PT}}^\dagger$  for (3.2) leads to the following ten constraints

$$\begin{aligned} \beta_1 &= 0, & \beta_2 &= 0, & \beta_3 &= \frac{\alpha_6}{2}, & \beta_4 &= -\frac{\alpha_5}{2}, & \beta_5 &= 0, \\ \beta_6 &= 0, & \beta_7 &= 0, & \beta_8 &= 0, & \beta_9 &= 0, & \beta_{10} &= -\frac{\theta\alpha_9}{2}, \end{aligned} \quad (3.3)$$

with all remaining constants left to be unrestricted. Thus provided we can compute the transformation from  $H_{\mathcal{PT}}$  to  $h_{\mathcal{PT}}$  in a well-defined manner, the solution of the ten equations (3.3) will characterize the domain in the parameter space, spanned by the constants  $\hat{c}_j$ , for which the eigenvalues of  $H_{\mathcal{PT}}$  are guaranteed to be real.

We assume here for this purpose the Dyson map to be linear in the deformed  $E_2$ -generators in the exponential

$$\eta = e^{\lambda J + \rho U + \tau V}, \quad \text{for } \lambda, \tau, \rho \in \mathbb{R}, \quad (3.4)$$

such that we can easily compute the adjoint action of this operator on the deformed  $E_2$ -generators. We obtain

$$\eta J \eta^{-1} = J + i(\rho V - \tau U) \frac{\sinh \lambda}{\lambda} + \left[ \rho U + \tau V + \frac{\theta}{\lambda}(\rho^2 + \tau^2) \right] \frac{1 - \cosh \lambda}{\lambda}, \quad (3.5)$$

$$\eta U \eta^{-1} = \left( U + \frac{\rho\theta}{\lambda} \right) \cosh \lambda - i \left( V + \frac{\tau\theta}{\lambda} \right) \sinh \lambda - \frac{\rho\theta}{\lambda}, \quad (3.6)$$

$$\eta V \eta^{-1} = \left( V + \frac{\tau\theta}{\lambda} \right) \cosh \lambda + i \left( U + \frac{\rho\theta}{\lambda} \right) \sinh \lambda - \frac{\tau\theta}{\lambda}. \quad (3.7)$$

As expected, we recover the expressions for the undeformed  $E_2$ -Lie algebraic reported in [1] in the limit  $\theta \rightarrow 0$ .

Notice that for Hermitian representations of  $J, U$  and  $V$  also  $\eta$  will also be Hermitian. Thus whenever the assumptions of this transformation are respected the eigenvalues of  $H_{\mathcal{PT}}$

must be real by construction. This means that when the constraints (3.3) allow  $\lambda, \tau, \rho \in \mathbb{R}$  we are in the  $\mathcal{PT}$ -symmetric regions in parameter space where  $H_{\mathcal{PT}}$  has real eigenvalues. In turn when the equations (3.3) impose  $\lambda, \tau, \rho \notin \mathbb{R}$  the  $\mathcal{PT}$ -symmetry is spontaneously broken and  $H_{\mathcal{PT}}$  acquires complex eigenvalues in form of conjugate pairs.

### 3.1 $\mathcal{PT}_5$ -symmetric Hamiltonians of deformed $E_2$ Lie algebraic type

We present here a detailed analysis of the  $\mathcal{PT}_5$ -symmetric non-Hermitian Hamiltonians, because these were also the models worked out in most detail for the undeformed version in [1]. The  $\mathcal{PT}_5$  invariant Hamiltonian

$$H_{\mathcal{PT}_5}(U, V, J) = \mu_1 J^2 + \mu_2 J + \mu_3 U + i\mu_4 V + \mu_5 UJ + i\mu_6 VJ + \mu_7 U^2 + \mu_8 V^2 + i\mu_9 UV. \quad (3.8)$$

is non-Hermitian unless  $\mu_6 = \mu_5 + 2\mu_4 = 0$ . In the undeformed case we found that the minimal requirement to satisfy the constraints (3.3) is

$$\tau = 0, \quad \rho = \frac{\lambda(\mu_5 - \mu_6 \coth \lambda)}{2\mu_1}, \quad \coth(2\lambda) = \frac{\mu_{78}}{\mu_{19}}, \quad \coth \lambda = \frac{\mu_{23}}{\mu_{24}}. \quad (3.9)$$

For convenience we introduced here the abbreviations

$$\mu_{78} := \frac{\mu_5^2 + \mu_6^2}{4\mu_1} - \mu_7 + \mu_8, \quad \mu_{19} := \frac{\mu_5 \mu_6}{2\mu_1} - \mu_9, \quad \mu_{23} := \frac{\mu_2 \mu_5}{2\mu_1} - \frac{\mu_6}{2} - \mu_3, \quad \mu_{24} := \frac{\mu_2 \mu_6}{2\mu_1} - \frac{\mu_5}{2} - \mu_4.$$

Thus, according to the last two constraints in (3.9), the domain in the parameter space for which  $H_{\mathcal{PT}_5}(U, V, J)$  is guaranteed to have real eigenvalues is characterized by the two inequalities

$$|\mu_{78}| \geq |\mu_{19}| \quad \text{and} \quad |\mu_{23}| \geq |\mu_{24}|. \quad (3.10)$$

Surprisingly when we carry out the analysis for the deformed algebra the first three constraints in (3.9) remain completely unchanged. The last one is replaced by

$$\mu_3 = \mu_3^{\theta=0} + \theta 2\mu_{56} \left[ \mu_{19} \left( \frac{1}{2} + \cosh \lambda \right) - \mu_{78} \sinh \lambda - \mu_{68} \frac{1 + \cosh \lambda}{\sinh \lambda} \right], \quad (3.11)$$

where we introduced

$$\mu_{56} := \frac{\mu_6 \cosh \lambda - \mu_5 \sinh \lambda}{2\mu_1(1 + \cosh \lambda)} \quad \text{and} \quad \mu_{68} := \frac{\mu_6^2}{4\mu_1} + \mu_8. \quad (3.12)$$

For brevity we have also denoted here by  $\mu_3^{\theta=0} = -\mu_{24} \coth \lambda + \mu_2 \mu_5 / (2\mu_1) - \mu_6 / 2$  the value of  $\mu_3$  as reported in [1], i.e. the value obtained from solving from the last constraint in (3.9) for  $\mu_3$ . This means the exceptional points resulting from the violation of the first inequality in (3.10) remain invariant under the deformation, whereas the one resulting from the second inequality acquires a  $\theta$ -dependence. It is clear from (3.11) that a generic discussion would be rather involved. Since the invariant exceptional point is less interesting, let us therefore make a special choice  $\mu_7 = (\mu_5^2 + \mu_6^2 + 4\mu_1 \mu_8) / (4\mu_1)$  and  $\mu_9 = \mu_5 \mu_6 / (2\mu_1)$  for which the third constraint in (3.9) no longer emerges. With this choice, equation (3.11) can be brought into the form

$$\coth \lambda = \frac{\mu_1 \mu_{23} + \theta \mu_5 \mu_{68}}{\mu_1 \mu_{24} + \theta \mu_6 \mu_{68}}. \quad (3.13)$$

The isospectral Hermitian Hamiltonian resulting from the similarity transformation with these constraints is then computed to

$$\begin{aligned} h_{\mathcal{PT}_5}(U, V, J) = & \mu_1 J^2 + (\mu_2 + \theta \mu_6 \mu_{56}) J + \left( \mu_8 + \frac{\mu_5^2}{4\mu_1} + \mu_6 \mu_{56} \right) U^2 + \mu_{68} V^2 \\ & + \left[ \frac{\mu_2}{\mu_1} \mu_{65} - \mu_{23} \cosh \lambda + \mu_{24} \sinh \lambda + \theta \mu_{56} \left( \frac{\mu_6}{\mu_1} \mu_{65} + 2\mu_{68} \sinh \lambda \right) \right] U \\ & + \mu_{65} \{U, J\} - \theta \left[ \frac{\mu_5 \mu_6}{4\mu_1} + \mu_{56} \left( \mu_{24} \cosh \lambda - \mu_{23} \sinh \lambda - \mu_4 - \frac{\mu_5}{2} \right) \right] \\ & - \theta^2 \mu_{56}^2 [\mu_{68}(1 + 2 \cosh \lambda) + \mu_8], \end{aligned} \quad (3.14)$$

where as usual  $\{A, B\} := AB + BA$  denotes the anti-commutator and

$$\mu_{65} := \frac{\mu_5}{2} - \frac{\mu_6}{2} \tanh \frac{\lambda}{2}. \quad (3.15)$$

By construction the eigenvalues of this Hamiltonian are real when the absolute value of the right hand side of (3.13) is greater than 1.

Making now the additional parameter choice

$$\mu_2 = 0, \quad \mu_5 = -2\mu_4, \quad \mu_6 = -2\mu_3, \quad \mu_8 = -\frac{\mu_3^2}{\mu_1}, \quad (3.16)$$

also the constraint (3.9) vanishes. This means the Dyson map is a well-defined transformation for all choices of the remaining free parameters. Thus we do not expect any  $\mathcal{PT}$ -symmetry breaking and all eigenvalues to be real. Indeed this is easily verified. Choosing furthermore  $\mu_3 = \mu_4 \coth(\lambda/2)$ , the isospectral Hermitian Hamiltonian simply results to

$$h_{\mathcal{PT}_5}(U, V, J) = \mu_1 J^2 + \varepsilon J + \frac{\theta \mu_4^2}{\mu_1} \left[ \varepsilon - \coth \left( \frac{\lambda}{2} \right) \right], \quad (3.17)$$

with  $\varepsilon = \theta \mu_4^2 / [\mu_1 \sinh^2(\lambda/2)]$ . Using the representation (2.3) in coordinate space with  $p_{x,y} = -i\hbar \partial_{x,y}$ , the Schrödinger equation  $h_{\mathcal{PT}_5} \psi = E \psi$  converts into

$$-\mu_1 \partial_\varphi^2 \psi(\varphi, r) + i\varepsilon \partial_\varphi \psi(\varphi, r) = E \psi(\varphi, r), \quad (3.18)$$

when using polar coordinates  $x = r \cos \varphi$  and  $y = r \sin \varphi$ . Equation (3.18) is easily solved to

$$\psi(\varphi, r) = c_1(r) e^{i(\varepsilon + \sqrt{\varepsilon^2 + 4E\mu_1})/(2\mu_1)\varphi} + c_2(r) e^{i(\varepsilon - \sqrt{\varepsilon^2 + 4E\mu_1})/(2\mu_1)\varphi}, \quad (3.19)$$

with  $c_1(r)$ ,  $c_2(r)$  being generic functions of  $r$  and arbitrary energies  $E$ . Quantizing now the system by demanding the periodicity  $\psi(\varphi + 2\pi, r) = \psi(\varphi, r)$  the energy discretises to

$$E_n = 4\pi^2 \mu_1 n^2 - \theta \frac{2\pi \mu_4^2}{\mu_1 \sinh^2(\lambda/2)} n. \quad (3.20)$$

As expected this energy is always real and no  $\mathcal{PT}$ -symmetry breaking can occur.

We have carried out a similar analysis for  $\mathcal{PT}_3$  and  $\mathcal{PT}_4$ -symmetric systems and found identical qualitative behaviour in the sense that the exceptional points governed by one of the inequalities remain invariant under the transformation and those characterized by a second inequality acquires a  $\theta$ -dependence.

## 4. Conclusion

We have introduced a natural and self-consistent deformation of the Euclidean Lie algebra  $E_2$ , by studying the effects of Bopp-shifts in some explicit representations. Of the previously [1] identified five different types of antilinear, e.g.  $\mathcal{PT}$ , symmetries for this algebra two of them were found to be broken by the deformation. For the remaining ones we studied invariant non-Hermitian systems expressed in terms of the generators of the deformed algebra (2.4). The main question addressed in this manuscript was to identify the domains in the parameter space where the non-Hermitian Hamiltonians possess real eigenvalues. This question was answered without the explicit computation of the eigenvalue spectrum by identifying instead the regions for which the Dyson map breaks down, which concretely meant that we derived a simply criterium for which the parameter  $\lambda$  in the transformation ceases to be real and becomes complex. The transition is characterized by the so-called exceptional points. We found two qualitatively different types of behaviour, one in which the exceptional points remain invariant under the deformation and one in which the corresponding energies acquire an explicit dependence on the deformation parameter  $\theta$ . For the  $\mathcal{PT}_5$ -symmetric Hamiltonian we provided a detailed analysis by successively fixing more and more of the free parameters, thus passing through the stages of having initially all types of exceptional points in the model, then having only the non-invariant ones and finally for a choice that eliminates all of the exceptional points and therefore excluding all possibilities of  $\mathcal{PT}$ -symmetry breaking.

Evidently there are many interesting open challenges left. Naturally one might investigate different types of deformations of the Euclidean Lie algebra  $E_2$  and also extend the analysis to higher ranks. In regard to the physical questions addressed in this context the precise nature of the breaking of the different  $\mathcal{PT}$ -symmetries would be interesting to investigate. Except for the simple model (3.17), we have circumvented here the explicit construction of the eigenvalue systems, but it would of course be valuable to establish whether the models considered are indeed solvable and confirm the spontaneous  $\mathcal{PT}$ -symmetry breaking also on the basis of the explicit eigenvalues.

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